

Perturbations of Cosmological and Black Hole Solutions in Massive gravity and Bi-gravity

Tsutomu Kobayashi,^a Masaru Siino,^b Masahide Yamaguchi,^b
Daisuke Yoshida^b

^aDepartment of Physics, Rikkyo University, Toshima, Tokyo 175-8501, Japan

^bDepartment of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan

E-mail: tsutomu@rikkyo.ac.jp, msiino@th.phys.titech.ac.jp,
gucci@phys.titech.ac.jp, yoshida@th.phys.titech.ac.jp

Abstract. We investigate perturbations of a class of spherically symmetric solutions in massive gravity and bi-gravity. The background equations of motion for the particular class of solutions we are interested in reduce to a set of the Einstein equations with a cosmological constant. Thus, the solutions in this class include all the spherically symmetric solutions in general relativity, such as the Friedmann-Lemaître-Robertson-Walker solution and the Schwarzschild (-de Sitter) solution, though the one-parameter family of two parameters of the theory admits such a class of solutions. We find that the equations of motion for the perturbations of this class of solutions also reduce to the perturbed Einstein equations at first and second order. Therefore, the stability of the solutions coincides with that of the corresponding solutions in general relativity. In particular, these solutions do not suffer from non-linear instabilities which often appear in the other cosmological solutions in massive gravity and bi-gravity.

Contents

1	Introduction	1
2	Review of bi-gravity	3
3	Bi-spherically symmetric background solutions	5
4	Linear perturbations	7
5	Second-order perturbations	8
6	Conclusions and discussion	9
A	Concrete examples of background solutions	10
A.1	Bi-cosmological solutions	10
A.2	Bi-Schwarzschild de Sitter solutions	11
B	Bianchi identity	12
C	Additional gauge symmetry of linear perturbations	13
D	Hamiltonian analysis of linear perturbations	15
D.1	Odd mode perturbations	15
D.2	Even mode perturbations	18

1 Introduction

Massive gravity is one of the potent candidates for modified theory of general relativity. As early as in 1939, a linear theory of massive gravity was proposed by Fierz and Pauli (FP) [1]. In order to avoid the inconsistency on massless limit of this theory [2, 3], a nonlinear extension of the FP theory was considered [4]. Boulware and Deser, however, found that the nonlinear theory simply extended from the FP theory contains an unphysical ghost degree of freedom (BD ghost) [5]. Because of this ghost problem, a healthy theory of non-linear massive gravity had not been established for a long time.

Recently, de Rham, Gabadadze, and Tolley (dRGT) proposed a mass potential which can remove the BD ghost mode in a decoupling limit [6, 7], and Hassan and Rosen have finally proven that the dRGT massive gravity theory is free from the BD ghost without taking the decoupling limit [8–10]. The dRGT theory of massive gravity has three parameters, graviton mass m and coupling constants of nonlinear self interactions, α_3, α_4 . Complementary approaches of the BD ghost problem are studied in refs. [11–14].

Massive gravity includes a non-dynamical tensor field called fiducial metric, $f_{\mu\nu}$, in order to construct a mass potential. For example, the FP and the original dRGT theories are constructed by adopting the Minkowski fiducial metric. Hassan and Rosen proposed an extended theory of dRGT massive gravity with a fiducial metric being dynamical as well by introducing the Einstein-Hilbert term of the fiducial metric in the action. They proved that

this theory is also free from the BD ghost [15]. Since this theory contains two symmetric dynamical tensor fields of metrics, it is called bi-gravity theory.

The tests of massive gravity and bi-gravity using cosmological and black hole solutions have been explored intensively. In dRGT massive gravity, several types of exact, homogeneous, and isotropic solutions have been known so far. One example is the open Friedmann-Lemaître-Robertson-Walker (FLRW) solution with the flat fiducial metric in the FLRW slice [16]. The second-order perturbations of this solution show nonlinear instability and hence this solution is not viable unfortunately [17–19]. It should be noted that similar solutions with an anisotropic fiducial metric are known to be stable [20–22]. Another example of cosmological solutions is that with a fiducial metric which is flat but expressed in terms of nontrivial coordinates. This class of solutions can be divided into two types. The first type includes the solutions found in refs. [23–25], which exist for the whole parameter region of α_3 and α_4 , while the other type includes the solutions found in refs. [26, 27], which exists only for a one parameter family in the parameter space (α_3, α_4) . Though the perturbations of the former type of solutions have already been studied in refs. [28–31], the perturbations of the latter type of solutions have not yet been investigated. Therefore, in this paper, we focus on the latter type of solutions. It should be noted that cosmological solutions with non-flat fiducial metrics are also studied in refs. [32–35].

The situations on cosmological solutions in bi-gravity are similar. The cosmological solutions found thus far are divided into two classes [36–39]. The first class is a solution with diagonal metric tensors [40–42]. The perturbations of this class of solutions have been studied intensively [43–55]. On the other hand, for the other class of a solution with off-diagonal components of physical or fiducial metric tensor, the perturbations have not yet been studied, similarly to the case of massive gravity. It should be noticed that coupling between matter and (bi-)metrics is a nontrivial issue in bi-gravity and is studied in refs. [56–63].

A lot of static and spherically symmetric solutions have also been found up to now. The classification of such spherically symmetric solutions is studied in refs. [39, 64, 65]. The exact Schwarzschild (-de Sitter) solutions are classified to the following three classes. The first class is a solution with diagonal metric tensors [66–69] and linear perturbations of this class of solutions are studied in ref. [69–73] in the framework of both massive gravity and bi-gravity. The second class of solutions is a solution with a off-diagonal metric tensor and arbitrary α_3 and α_4 [74], and the perturbation of this solution is studied in ref. [69, 75]. The last class is a solution with a off-diagonal metric tensor and a special choice of the parameters α_3 and α_4 [76–78], where linear perturbations have been studied only in massive gravity with a flat fiducial metric [79] and not in bi-gravity.

In the present article, we will give a unified and general treatment for solutions with an off-diagonal metric tensor in massive gravity and bi-gravity belonging to a one parameter family of α_3 and α_4 , which include both cosmological [26, 27] and spherically symmetric black hole solutions [64, 76–79]. We will find that the equations of motion for this class of solutions exactly reduce to those of general relativity (GR) with a cosmological constant not only at the background and linear (first-order) perturbation level but also at the level of quadratic (second-order) perturbations. This result shows that massive gravity and bi-gravity can allow any spherically symmetric solution of GR including its stability, the evolution of linear perturbations, and the backreaction from linear perturbations, while it simultaneously implies that one cannot distinguish massive gravity or bi-gravity from GR by using spherically symmetric solutions and their perturbations at least up to quadratic order.

Our paper is organized as follows. In the next section, we briefly review the theory

of bi-gravity (and massive gravity as a trivial case of a fixed fiducial metric) and derive the equations of motion in a general setting. In section 3, we derive a generic non-diagonal spherically symmetric background solution. Then, we investigate linear perturbations around those background solutions in section 4. There we will see that the terms coming from the mass potential must vanish in order to satisfy the Bianchi identity. In section 5, we investigate higher order perturbations and find that the same results as the linear perturbations apply for the quadratic perturbations. The final section is devoted to summary and discussion. Some details will be given in the appendices.

2 Review of bi-gravity

In this section, we give a brief review of bi-gravity. Bi-gravity is a theory consisting of two dynamical tensor fields, $g_{\mu\nu}$ and $f_{\mu\nu}$, called physical and fiducial metrics, respectively. Massive gravity can be understood as a special case where the fiducial metric is fixed and non-dynamical. Its action is given by the Einstein-Hilbert term for each metric with the interaction term S_{mass} and matter actions:

$$S = \frac{1}{2}M_{\text{pl}}^2 \int d^4x \sqrt{-g} R[g] + \frac{1}{2}\kappa^2 M_{\text{pl}}^2 \int d^4x \sqrt{-f} R[f] + S_{\text{mass}}[g, f] + S_{\text{matter}}[g] + S_{\text{matter}}[f], \quad (2.1)$$

where $R[\cdot]$ is the Ricci scalar and κ represents the ratio of the effective Planck masses for $g_{\mu\nu}$ and $f_{\mu\nu}$. In the case of $\kappa = 0$, the tensor field $f_{\mu\nu}$ does not have its kinetic term and hence is non-dynamical. This case corresponds to the massive gravity theory originally proposed by de Rham, Gabadadze, and Tolley [6, 7]. $S_{\text{matter}}[g]$ and $S_{\text{matter}}[f]$ are the matter actions coupled to g and f , respectively,

$$S_{\text{matter}}[g] = \int d^4x \sqrt{-g} \mathcal{L}_{\text{matter}}^{(g)}, \quad (2.2)$$

$$S_{\text{matter}}[f] = \int d^4x \sqrt{-f} \mathcal{L}_{\text{matter}}^{(f)}. \quad (2.3)$$

Here we implicitly assume the matter actions possess the two general covariance with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$ separately, though the full theory does not have such a symmetry. It should be noted that another type of matter coupling, which does not possess the two general covariance, is also studied by [56–63]. The energy-momentum tensors coming from $S_{\text{matter}}[g]$ and $S_{\text{matter}}[f]$ are defined as

$$T_{(g)}^{\mu}{}_{\nu} = \frac{2}{\sqrt{-g}} g^{\mu\rho} \frac{\delta S_{\text{matter}}[g]}{\delta g^{\rho\nu}}, \quad (2.4)$$

$$T_{(f)}^{\mu}{}_{\nu} = -\frac{2}{\sqrt{-f}} \frac{\delta S_{\text{matter}}[f]}{\delta f_{\mu\rho}} f_{\rho\nu}. \quad (2.5)$$

Due to the two general covariance of matter actions we assumed, both energy-momentum tensors are conserved, that is, $\nabla_{\mu}^{(g)} T_{(g)}^{\mu}{}_{\nu} = 0$ and $\nabla_{\mu}^{(f)} T_{(f)}^{\mu}{}_{\nu} = 0$, where $\nabla_{\mu}^{(g)}$ and $\nabla_{\mu}^{(f)}$ are the covariant derivatives with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, respectively. Hereafter, we will omit the suffixes g and f when no confusion is expected.

Now the interaction term S_{mass} is tuned to be free from the BD ghost mode and given by

$$S_{\text{mass}}[g, f] = \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g} 2m^2 \sum_{i=0}^4 \beta_i e_i(\gamma), \quad (2.6)$$

where i -th order contributions $e_i(\gamma)$ are given by

$$e_0(\gamma) = 1, \quad (2.7)$$

$$e_1(\gamma) = \text{Tr}[\gamma], \quad (2.8)$$

$$e_2(\gamma) = \frac{1}{2} (\text{Tr}[\gamma]^2 - \text{Tr}[\gamma^2]), \quad (2.9)$$

$$e_3(\gamma) = \frac{1}{3!} (\text{Tr}[\gamma]^3 - 3\text{Tr}[\gamma]\text{Tr}[\gamma^2] + 2\text{Tr}[\gamma^3]), \quad (2.10)$$

$$e_4(\gamma) = \det(\gamma), \quad (2.11)$$

with

$$\gamma^\mu{}_\nu = (\sqrt{g^{-1}}f)^\mu{}_\nu, \quad (2.12)$$

and m, β_i being free parameters of the interaction term. Since they always appear in the combination $m^2\beta_i$, essentially there are five free parameters. The space of parameters corresponds to the one of the three parameters of the dRGT theory, m, α_3, α_4 , and two cosmological constants, $\Lambda^{(g)}, \Lambda^{(f)}$, and their relations are given by¹

$$m^2\beta_0 = -\Lambda^{(g)} + m^2(6 + 4\alpha_3 + \alpha_4), \quad (2.13)$$

$$m^2\beta_1 = m^2(-3 - 3\alpha_3 - \alpha_4), \quad (2.14)$$

$$m^2\beta_2 = m^2(1 + 2\alpha_3 + \alpha_4), \quad (2.15)$$

$$m^2\beta_3 = m^2(-\alpha_3 - \alpha_4), \quad (2.16)$$

$$m^2\beta_4 = -\kappa^2\Lambda^{(f)} + m^2\alpha_4. \quad (2.17)$$

Taking the variation of the action with respect to $g_{\mu\nu}$ and $f_{\mu\nu}$, we will obtain the equations of motion for the two tensor fields. The equations of motion for $g_{\mu\nu}$ are given by

$$G[g]^\mu{}_\nu + X_{(g)}^\mu{}_\nu = \frac{1}{M_{\text{pl}}^2} T_{(g)}^\mu{}_\nu, \quad (2.18)$$

where $G[g]^\mu{}_\nu$ is the Einstein tensor constructed from $g_{\mu\nu}$ and

$$X_{(g)}^\mu{}_\nu = 2m^2 \left(\tau^\mu{}_\nu - \frac{1}{2} \delta^\mu_\nu \sum_{i=0}^3 \beta_i e_i(\gamma) \right), \quad (2.19)$$

$$\begin{aligned} \tau^\mu{}_\nu &= \frac{1}{2} [\beta_1 \gamma^\mu{}_\nu + \beta_2 (e_1(\gamma) \gamma^\mu{}_\nu - (\gamma^2)^\mu{}_\nu) \\ &\quad + \beta_3 (e_2(\gamma) \gamma^\mu{}_\nu - e_1(\gamma) (\gamma^2)^\mu{}_\nu + (\gamma^3)^\mu{}_\nu)]. \end{aligned} \quad (2.20)$$

¹ It is useful to rewrite the action in term of these parameters as follows,

$$S_{\text{mass}}[g, f] = \frac{M_{\text{pl}}^2}{2} \int d^4x \left[\sqrt{-g} 2m^2 \sum_{i=2}^4 \alpha_i e_i(\mathcal{K}) + \sqrt{-g} (-2\Lambda^{(g)}) + \sqrt{-f} (-2\kappa^2 \Lambda^{(f)}) \right],$$

with $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \gamma^\mu{}_\nu$ and $\alpha_2 = 1$.

The indices here are raised or lowered by $g_{\mu\nu}$. The equations of motion for $f_{\mu\nu}$ are given by

$$G[f]^\mu{}_\nu + X_{(f)}^\mu{}_\nu = \frac{1}{\kappa^2 M_{\text{pl}}^2} T_{(f)}^\mu{}_\nu, \quad (2.21)$$

where $G[f]^\mu{}_\nu$ is the Einstein tensor constructed from $f_{\mu\nu}$ and

$$X_{(f)}^\mu{}_\nu = -\frac{m^2}{\kappa^2} \text{sgn}(\det \gamma) \left(\frac{2}{\det \gamma} \tau^\mu{}_\nu + \beta_4 \delta^\mu{}_\nu \right). \quad (2.22)$$

The indices here are raised or lowered by $f_{\mu\nu}$.

3 Bi-spherically symmetric background solutions

Here, we attempt to classify some of the spherically symmetric solutions in bi-gravity and identify those which obey the same equations of motion as in general relativity. These classes of solutions include the cosmological and black hole solutions known so far [26, 36–39, 64, 79].

Let us consider the following bi-spherically symmetric metrics:

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = \bar{g}_{00}(t, r) dt^2 + 2\bar{g}_{01}(t, r) dt dr + \bar{g}_{11}(t, r) dr^2 + R(t, r)^2 d\Omega^2, \quad (3.1)$$

$$\bar{f}_{\mu\nu} dx^\mu dx^\nu = \bar{f}_{00}(t, r) dt^2 + 2\bar{f}_{01}(t, r) dt dr + \bar{f}_{11}(t, r) dr^2 + A^2(t, r) R^2(t, r) d\Omega^2, \quad (3.2)$$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. The matrix $\bar{g}^{-1}\bar{f}$ takes the following form,

$$(\bar{g}^{-1}\bar{f})^\mu{}_\nu = \begin{pmatrix} (\bar{g}^{-1}\bar{f})^0{}_0 & (\bar{g}^{-1}\bar{f})^0{}_1 & 0 & 0 \\ (\bar{g}^{-1}\bar{f})^1{}_0 & (\bar{g}^{-1}\bar{f})^1{}_1 & 0 & 0 \\ 0 & 0 & A^2(t, r) & 0 \\ 0 & 0 & 0 & A^2(t, r) \end{pmatrix}, \quad (3.3)$$

and from these ansatz it is straightforward to see that the square root of the above matrix is of the form

$$\bar{\gamma}^\mu{}_\nu = \left(\sqrt{\bar{g}^{-1}\bar{f}} \right)^\mu{}_\nu = \begin{pmatrix} a(t, r) & b(t, r) & 0 & 0 \\ c(t, r) & d(t, r) & 0 & 0 \\ 0 & 0 & A(t, r) & 0 \\ 0 & 0 & 0 & A(t, r) \end{pmatrix}. \quad (3.4)$$

It should be emphasized that the following discussion does not rely on the concrete expressions of $a(t, r)$, $b(t, r)$, $c(t, r)$, and $d(t, r)$, but rather relies only on the fact that $\bar{\gamma}^\mu{}_\nu$ is of the form of eq. (3.4).

As explained earlier, we are interested in the case where the equations of motion for both metrics reduce to the Einstein equations with cosmological constants at the background level. Therefore, in order for $X_{(g)}^\mu{}_\nu$ to be a cosmological term, the non-trivial off-diagonal components,

$$\bar{X}_{(g)}^0{}_1 = -m^2 b[3 - 2A + (A - 3)(A - 1)\alpha_3 + (A - 1)^2\alpha_4], \quad (3.5)$$

$$\bar{X}_{(g)}^1{}_0 = -m^2 c[3 - 2A + (A - 3)(A - 1)\alpha_3 + (A - 1)^2\alpha_4], \quad (3.6)$$

must vanish. We focus on the case of $b(t, r) \neq 0$ or $c(t, r) \neq 0$, and $A \neq 1$, since with $b(t, r) = c(t, r) = 0$, we will obtain a diagonal metrics as mentioned in Sec. 1 and the perturbations of such diagonal solutions have already been studied.

For non-diagonal solutions we are interested in, the condition that eqs. (3.5) and (3.6) vanish leads to

$$A(t, r) = \frac{2\alpha_3 + \alpha_4 + 1 \pm \sqrt{\alpha_3^2 + \alpha_3 - \alpha_4 + 1}}{\alpha_3 + \alpha_4} = \text{const.} \quad (3.7)$$

Another requirement necessary for $\bar{X}_{(g)}{}^\mu{}_\nu$ to be a cosmological term is

$$\bar{X}_{(g)}{}^0{}_0 - \bar{X}_{(g)}{}^2{}_2 = (1 - A)C(t, r) [A - 2 + (A - 1)\alpha_3] = 0, \quad (3.8)$$

where, we have defined $C(t, r)$ as

$$C(t, r) = m^2 \frac{A^2 - A(a + d) + ad - bc}{(1 - A)^2}. \quad (3.9)$$

With $C(t, r) = 0$, at least three eigenvalues of $\bar{\gamma}^\mu{}_\nu$ are equal to A . This class of solutions includes the cosmological solutions found in refs. [23–25] and the Schwarzschild solutions obtained in ref. [75]. The perturbations of those solutions have already been studied in detail in refs. [28–31].

In the present study, we therefore concentrate on the case with $C(t, r) \neq 0$. In this case, the solution to eq. (3.8) is

$$A = \frac{2 + \alpha_3}{1 + \alpha_3}. \quad (3.10)$$

Here we have assumed that $\alpha_3 \neq -1$. Equations (3.7) and (3.10) are consistent provided that the parameters of the theory, α_3 and α_4 , satisfy

$$0 = 1 + \alpha_3 + \alpha_3^2 - \alpha_4 \quad (= \beta_2^2 - \beta_1\beta_3). \quad (3.11)$$

This is equivalent to the condition that the two branches of the solution (3.7) degenerate. Thus, we see that only the particular one-parameter family of α_3 and α_4 satisfying (3.11) admits the class of solutions we are focusing on. Note that when eq. (3.11) is fulfilled A can also be expressed simply as $A = -\beta_2/\beta_3$. Note also that the range of α_4 is limited as $\alpha_4 = (\alpha_3 + 1/2)^2 + 3/4 \geq 3/4$.

In this one-parameter family of α_3 and α_4 with eq. (3.11), the interaction terms for bi-spherically symmetric metrics (3.1) and (3.2) with eq. (3.10) are of the form of a cosmological term. For $g_{\mu\nu}$ the interaction term gives

$$\bar{X}_{(g)}{}^\mu{}_\nu = \Lambda_{\text{eff}}^{(g)} \delta^\mu{}_\nu, \quad (3.12)$$

with

$$\Lambda_{\text{eff}}^{(g)} = m^2(A - 1) + \Lambda^{(g)}, \quad (3.13)$$

while for $f_{\mu\nu}$

$$\bar{X}_{(f)}{}^\mu{}_\nu = \Lambda_{\text{eff}}^{(f)} \delta^\mu{}_\nu, \quad (3.14)$$

$$\Lambda_{\text{eff}}^{(f)} = \text{sgn}(ad - bc) \left(-\frac{m^2}{\kappa^2} \frac{A - 1}{A} + \Lambda^{(f)} \right). \quad (3.15)$$

In the case of dRGT massive gravity, $f_{\mu\nu}$ is not a dynamical but a fixed metric, and hence we need not consider the equations of motion for $f_{\mu\nu}$.

The class of solutions with $C(t, r) \neq 0$ includes the cosmological solutions [26, 27], the black hole solutions [76–79], the Lemaître-Tolman-Bondi (LTB) solution [27], and the Reissner-Nordström (RN) solution [77]. Since the equations of motion for $g_{\mu\nu}$ (and, in fact, those for $f_{\mu\nu}$ as well) reduce to the Einstein equations with a cosmological constant, any spherically symmetric solution in GR is also a solution of the one-parameter subclass (3.11) of bi-gravity and massive gravity with a suitable fiducial metric. In appendix A, we present some examples of bi-FLRW and bi-Schwarzschild-de Sitter solutions belonging to this class.

4 Linear perturbations

Now, we analyze linear perturbations around bi-spherically symmetric solutions given in the previous section. The two tensor fields of metrics are perturbed as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad (4.1)$$

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu}. \quad (4.2)$$

The first-order perturbation, $\delta\gamma^\mu{}_\nu$, of $\gamma^\mu{}_\nu$ is defined as

$$\sqrt{g^{-1}}f = \gamma^\mu{}_\nu = \bar{\gamma}^\mu{}_\nu + \delta\gamma^\mu{}_\nu + \mathcal{O}(\text{second-order perturbations}), \quad (4.3)$$

which can be written in terms of the metric perturbations by solving the following equations,

$$\bar{\gamma}^\mu{}_\rho \delta\gamma^\rho{}_\nu + \delta\gamma^\mu{}_\rho \bar{\gamma}^\rho{}_\nu = -\delta g^\mu{}_\rho \bar{\gamma}^\rho{}_\sigma \bar{\gamma}^\sigma{}_\nu + \delta f^\mu{}_\nu, \quad (4.4)$$

where $\delta g^\mu{}_\nu = \bar{g}^{\mu\rho} \delta g_{\rho\nu}$ and $\delta f^\mu{}_\nu = \bar{g}^{\mu\rho} \delta f_{\rho\nu}$. For our purpose we do not need the explicit form of the solution to the above equation, though it is obtained for a general fiducial metric in ref. [80]. Actually, without the explicit form of $\delta\gamma^\mu{}_\nu$, we can directly calculate $X^\mu{}_\nu$ from eq. (4.3) with eq. (3.4) as

$$\delta X_{(g)}{}^\mu{}_\nu = C(t, r) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta\gamma^3{}_3 & -\delta\gamma^2{}_3 \\ 0 & 0 & -\delta\gamma^3{}_2 & \delta\gamma^2{}_2 \end{pmatrix}. \quad (4.5)$$

Since the Einstein tensor satisfies the Bianchi identity $\nabla_\mu^{(g)} G^\mu{}_\nu[g] = 0$ and the energy-momentum tensor is conserved, the tensor $X_{(g)}{}^\mu{}_\nu$ also satisfies $\nabla_\mu X_{(g)}{}^\mu{}_\nu = 0$. As demonstrated in appendix B, this requirement leads to a stronger condition

$$\delta X_{(g)}{}^\mu{}_\nu = 0, \quad (4.6)$$

which yields $\delta\gamma^a{}_b = 0$ for $a, b = 2, 3$. (Note that we are interested in the case with $C(t, r) \neq 0$.) Since

$$\delta X_{(f)}{}^\mu{}_\nu = -\frac{1}{\kappa^2 A^2 |ad - bc|} \delta X_{(g)}{}^\mu{}_\nu, \quad (4.7)$$

eq. (4.6) also implies $\delta X_{(f)}{}^\mu{}_\nu = 0$. Thus, the equations of motion for the linear perturbations $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ reduce to the linearized Einstein equations.

In order to see the implications of the equation (4.6) in more detail, we express $\delta\gamma^\mu{}_\nu$ in terms of the metric perturbations. Since only the angular components of $\delta\gamma^\mu{}_\nu$ enter the equation (4.6), we only have to deal with the angular components of the equation (4.4), which can easily be solved because $\bar{\gamma}^a{}_b = A\delta^a{}_b$ for $a, b = 2, 3$. In fact, eq. (4.4) reduces to

$$2A\delta\gamma^a{}_b = -A^2\delta g^a{}_b + \delta f^a{}_b = 0. \quad (4.8)$$

To sum up, the equations of motion for the first-order perturbations are equivalent to the following three equations:

$$\delta G[g]^\mu{}_\nu = \delta T_{(g)}^\mu{}_\nu, \quad (4.9)$$

$$\delta G[f]^\mu{}_\nu = \delta T_{(f)}^\mu{}_\nu, \quad (4.10)$$

$$A^2\delta g_{ab} - \delta f_{ab} = 0. \quad (4.11)$$

This is one of the main results of this investigation. The equations of motion for the perturbations of the two metrics coincide with the perturbed Einstein equations, though $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ are subject to eq. (4.11).

Then let us count the number of graviton degrees of freedom for this perturbed system. Each symmetric tensor field of metric has ten components, and there are, respectively, four constraints (the Hamiltonian and momentum constraints) in eqs. (4.9) and (4.10), since those equations are the same as the perturbed Einstein equations. Furthermore, eq. (4.11) gives three constraints among the angular components of the perturbed metrics. We have four spacetime coordinates and hence there are four gauge degrees of freedom representing the choice of coordinates. In addition to those familiar gauge degrees of freedom, it turns out that there still remains another gauge transformation retaining the equations of motion (4.9), (4.10), and (4.11), as explicitly shown in appendix C. Note that this gauge degree of freedom corresponds to the ambiguity of the linear perturbations mentioned in ref. [79] for the Schwarzschild-de Sitter solution in the dRGT theory. Thus, the number of the remaining degrees of freedom is $10 \times 2 - 4 \times 2 - 3 - (4 + 1) = 4$, which coincides with that of two massless gravitons. We can confirm that this is consistent with the result of the Hamiltonian analysis given in appendix D: there are ten first class constraints and twelve second class constraints, and hence there are 8 ($= 40 - 10 \times 2 - 12$) degrees of freedom in phase space.

The above analysis can be applied to dRGT massive gravity only with eqs. (4.9) and (4.11) because the derivations of these equations do not depend on the equation of motion for $f_{\mu\nu}$. Since, in this case, $\delta f_{\mu\nu}$ is composed of Stückelberg fields, the condition (4.11) just determines perturbations of Stückelberg fields. The remaining variables $\delta g_{\mu\nu}$ are governed by the Einstein equations and additional gauge symmetry appears as gauge degree of freedom for Stückelberg fields.

5 Second-order perturbations

In the previous section, we have shown that the first-order perturbations obey the perturbed Einstein equations and hence the behavior of the perturbations coincides with that of GR, though $\delta g_{\mu\nu}$ and $\delta f_{\mu\nu}$ are subject to eq. (4.11). One may then ask the question as to how one can discriminate this class of solutions in bi-gravity from the corresponding solutions in GR.

One possibility is to take into account the back reaction on the physical metric $g_{\mu\nu}$ from $\delta f_{\mu\nu}$ at second order. For this purpose, we incorporate second-order perturbations as follows:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} + g^{(2)}_{\mu\nu}, \quad (5.1)$$

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu} + f^{(2)}_{\mu\nu}. \quad (5.2)$$

The perturbed metrics now give rise to the second-order perturbations of $\gamma^\mu{}_\nu$ as

$$\gamma^\mu{}_\nu = \bar{\gamma}^\mu{}_\nu + \delta\gamma^\mu{}_\nu + \gamma^{(2)\mu}{}_\nu + \mathcal{O}(\text{third-order perturbations}), \quad (5.3)$$

where $\bar{\gamma}^\mu{}_\nu$ is the background quantity defined in eq. (3.4) and $\delta\gamma^\mu{}_\nu$ satisfies eq. (4.6), and hence $\delta\gamma^a{}_b = 0$ for $a, b = 2, 3$. The interaction term in the equations of motion for $g_{\mu\nu}$ can be calculated explicitly even at second order, and is given by

$$X_{(g)}^{(2)\mu}{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X^{(2)2}{}_2(t, r, \theta, \phi) & X^{(2)2}{}_3(t, r, \theta, \phi) \\ 0 & 0 & \frac{X^{(2)2}{}_3(t, r, \theta, \phi)}{\sin^2 \theta} & X^{(2)3}{}_3(t, r, \theta, \phi) \end{pmatrix}. \quad (5.4)$$

This tensor satisfies the conditions assumed in appendix B, which, together with the Bianchi identity, yield

$$X_{(g)}^{(2)\mu}{}_\nu = 0. \quad (5.5)$$

Even at second order, $X_{(g)}^{(2)\mu}{}_\nu$ is proportional to $X_{(f)}^{(2)\mu}{}_\nu$,

$$X_{(f)}^{(2)\mu}{}_\nu = -\frac{1}{\kappa^2 A^2 |ad - bc|} X_{(g)}^{(2)\mu}{}_\nu, \quad (5.6)$$

leading to $X_{(f)}^{(2)\mu}{}_\nu = 0$ as well. These conditions provide the relation between $g^{(2)}_{ab}$ and $f^{(2)}_{ab}$ as follows:

$$\gamma^{(2)a}{}_b = -\frac{1}{A^2 - A(a + d) + ad - bc} \delta\gamma^a{}_A (\bar{\gamma}^A{}_B - (\bar{\gamma}^C{}_C - A) \delta^A{}_B) \delta\gamma^B{}_b, \quad (5.7)$$

for $a, b = 2, 3$ and $A, B, C = 0, 1$. Thus, the metric perturbations obey the perturbed Einstein equations also at second order, and the number of graviton degrees of freedom coincides with that of two massless gravitons even at second order.

This fact indicates that one cannot discriminate this class of solutions from the corresponding solutions in GR even at second order, unfortunately. On the other hand, this fact, fortunately, implies that our solutions are free from non-linear instabilities even in cubic action, which plague many cosmological solutions in massive gravity, such as the diagonal open FLRW solution [16–18], flat FLRW solution [24, 28], and de Sitter solution [23, 28].

6 Conclusions and discussion

In the present study we have investigated the perturbations of a class of spherically symmetric solutions in massive gravity and bi-gravity. First, we classified spherically symmetric solutions in massive gravity and bi-gravity and identified the specific class for which the background equations of motion are identical to a set of the Einstein equations with a cosmological

constant. These solutions are allowed only with the one-parameter family of α_3 and α_4 satisfying eqs. (3.11). This class of solutions includes many known solutions, e.g., the FLRW solutions in ref. [26, 27], the Schwarzschild(-de Sitter) solutions in ref. [76–79], the LTB solution in ref. [27], and the RN solution in ref. [77]. In fact, any spherically symmetric solution in GR is included in this class with a suitable choice of the fiducial metric $f_{\mu\nu}$.

Next, we have investigated linear perturbations on this class of solutions. We have found that the interaction terms in the equations of motion for both metrics, $\delta X_{(g)}^\mu{}_\nu$ and $\delta X_{(f)}^\mu{}_\nu$, vanish thanks to the Bianchi identities, and hence the equations of motion reduce to eqs. (4.9)–(4.11), which are the perturbed Einstein equations with the relation (4.11).

We have also found that, in addition to the usual gauge symmetry associated with spacetime coordinate transformation, there is another gauge symmetry of the linear perturbations given by eqs. (C.8)–(C.11), which has already been known for the perturbations of the Schwarzschild de Sitter solution in dRGT massive gravity [79].

We have shown that the above result applies to second-order perturbations as well. Thus, one cannot distinguish this class of solutions in massive gravity and bi-gravity from the corresponding solutions of GR up to second order. This fact, however, implies that this class of solutions do not suffer from the non-linear instabilities, which often appear in the other cosmological solutions in massive gravity and bi-gravity. These aspects would suggest that massive gravity or bi-gravity with this one parameter family in (α_3, α_4) may have additional fully non-linear symmetry, which may be responsible for the stability. Further investigations are necessary to clarify this point.

In this article, only spherically symmetric background solutions are discussed. So, it is an interesting and open question whether the results obtained in this article hold for more general background solutions. Our analysis on the background solutions can at least be applied to any $\bar{\gamma}$ having the form of eq. (3.4) in any basis vectors, because the background equations of motion (3.12) can be obtained in an algebraic way from eq. (3.4) irrespective of a concrete expression for $\bar{\gamma}$. Extending the above analysis to linear and non-linear perturbations of more general solutions, however, is a non-trivial issue simply because such analysis accompanies the derivatives. We will address these issues in a future publication.

Acknowledgments

This work was in part supported by the JSPS Grant-in-Aid for Scientific Research Nos. 24740161 (T.K.), 25287054 (M.Y.), 26610062 (M.Y.), the JSPS Grant-in-Aid for Scientific Research on Innovative Areas No. 15H05888 (T.K. and M.Y.), and the JSPS Research Fellowship for Young Scientists, No. 26-11495 (D.Y.).

A Concrete examples of background solutions

A.1 Bi-cosmological solutions

First we consider a family of bi-FLRW solutions, in which physical metric takes the following FLRW form:

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right]. \quad (\text{A.1})$$

Comparing this metric with eq. (3.1) yields $R(t, r) = a(t)r$. We assume that $f_{\mu\nu}$ takes the same FLRW metric but in a coordinate $(\tilde{t}(t, r), \tilde{r}(t, r), \theta, \phi)$ different from that of $g_{\mu\nu}$,

$$\begin{aligned} f_{\mu\nu}dx^\mu dx^\nu &= -d\tilde{t}^2 + b^2(\tilde{t}) \left[\frac{d\tilde{r}^2}{1 - \tilde{K}\tilde{r}^2} + \tilde{r}^2 d\Omega^2 \right] \\ &= f_{00}dt^2 + 2f_{01}dtdr + f_{11}dr^2 + b^2(\tilde{t})\tilde{r}^2(t, r)d\Omega^2, \end{aligned} \quad (\text{A.2})$$

where

$$f_{00} = -\left(\frac{\partial\tilde{t}}{\partial t}\right)^2 + \frac{b^2(\tilde{t}(t, r))}{1 - \tilde{K}\tilde{r}^2(t, r)} \left(\frac{\partial\tilde{r}}{\partial t}\right)^2, \quad (\text{A.3})$$

$$f_{01} = -\frac{\partial\tilde{t}}{\partial t}\frac{\partial\tilde{t}}{\partial r} + \frac{b^2(\tilde{t}(t, r))}{1 - \tilde{K}\tilde{r}^2(t, r)} \frac{\partial\tilde{t}}{\partial t}\frac{\partial\tilde{r}}{\partial r}, \quad (\text{A.4})$$

$$f_{11} = -\left(\frac{\partial\tilde{t}}{\partial r}\right)^2 + \frac{b^2(\tilde{t}(t, r))}{1 - \tilde{K}\tilde{r}^2(t, r)} \left(\frac{\partial\tilde{r}}{\partial r}\right)^2. \quad (\text{A.5})$$

In order to apply the results of the main body, the radial coordinate \tilde{r} is determined to satisfy the following relation,

$$\tilde{r}(t, r) = \frac{AR(t, r)}{b(\tilde{t}(t, r))} = A\frac{a(t)r}{b(\tilde{t}(t, r))}, \quad (\text{A.6})$$

while the time coordinate \tilde{t} is arbitrary. In this case, the equations of motion for both metrics become Einstein equations with cosmological constants so that $a(t)$ and $b(\tilde{t})$ obey the Friedmann equation with respect to each (cosmic) time, t or \tilde{t} . This kind of bi-FLRW solution becomes a slight generalization of that found in ref. [36], in which a specific choice of the coordinate \tilde{t} is adopted.

For $b = 1$, $\tilde{K} = 0$, and $\Lambda_{\text{eff}}^{(f)} = 0$, the fiducial metric $f_{\mu\nu}$ becomes the flat Minkowski one and hence this bi-cosmological solution includes that obtained in ref. [26, 27] in dRGT massive gravity with the flat fiducial metric.

A.2 Bi-Schwarzschild de Sitter solutions

Our results are applied to the following bi-Schwarzschild de Sitter metrics as well:

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{r_{(g)}}{r} + \Lambda_{\text{eff}}^{(g)}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{r_{(g)}}{r} + \Lambda_{\text{eff}}^{(g)}r^2} + r^2d\Omega^2, \quad (\text{A.7})$$

$$f_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{\tilde{r}_{(f)}}{\tilde{r}} + \Lambda_{\text{eff}}^{(f)}\tilde{r}^2\right)d\tilde{t}^2 + \frac{d\tilde{r}^2}{1 - \frac{\tilde{r}_{(f)}}{\tilde{r}} + \Lambda_{\text{eff}}^{(f)}\tilde{r}^2} + \tilde{r}^2d\Omega^2 \quad (\text{A.8})$$

with

$$\tilde{r} = Ar, \quad (\text{A.9})$$

where $r_{(g)}$ and $\tilde{r}_{(f)}$ represent Schwarzschild radii, and $\Lambda_{\text{eff}}^{(g)}$ and $\Lambda_{\text{eff}}^{(f)}$ are effective cosmological constants defined in eqs. (3.13) and (3.15), respectively. Since the Schwarzschild-de Sitter metric is a solution of Einstein equation with cosmological constant, this is a vacuum solution in our setting. As is the case with the cosmological solution, this black hole solution can be obtained with arbitrary choice of the time coordinate $\tilde{t}(t, r)$. By tuning the parameters β_0 and β_4 , we can set $\Lambda_{\text{eff}}^{(g)}$ and $\Lambda_{\text{eff}}^{(f)}$ to be zeros simultaneously, which corresponds to a bi-Schwarzschild solution.

B Bianchi identity

In this appendix, we will show that a symmetric tensor satisfying a condition given below must vanish as long as it obeys Bianchi identity and the background metric $\bar{g}_{\mu\nu}$ takes the matrix form of eq. (3.1).

Let us consider the following symmetric tensor $X_{\mu\nu}$:

$$X^\mu{}_\nu = \Lambda \delta^\mu_\nu + \epsilon^n X^{(n)\mu}{}_\nu + \mathcal{O}(\epsilon^{n+1}) \quad (\text{B.1})$$

with

$$X^{(n)0}{}_\mu = X^{(n)1}{}_\mu = X^{(n)\mu}{}_0 = X^{(n)\mu}{}_1 = 0, \quad (\text{B.2})$$

where ϵ denotes the order of perturbations and Λ is a constant. The goal of this section is to show that $X^{(n)\mu}{}_\nu$ vanishes if the Bianchi identity, $\nabla_\mu X^\mu{}_\nu = 0$, is imposed.

The tensor $\bar{g}_{\mu\rho} X^{(n)\rho}{}_\nu$ is symmetric because

$$X_{\mu\nu} = g_{\mu\rho} X^\rho{}_\nu \quad (\text{B.3})$$

$$= \Lambda g_{\mu\nu} + \bar{g}_{\mu\rho} X^{(n)\rho}{}_\nu + \mathcal{O}(\epsilon^{n+1}), \quad (\text{B.4})$$

and both of $X_{\mu\nu}$ and $\Lambda g_{\mu\nu}$ are symmetric. Then, from the property of the background metric $\bar{g}_{\mu\nu}$, it is characterized by three arbitrary functions as follows:

$$\bar{g}_{\mu\rho} X^{(n)\rho}{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X_{22}^{(n)}(t, r, \theta, \phi) & X_{23}^{(n)}(t, r, \theta, \phi) \\ 0 & 0 & X_{23}^{(n)}(t, r, \theta, \phi) & X_{33}^{(n)}(t, r, \theta, \phi) \end{pmatrix}, \quad (\text{B.5})$$

or equivalently,

$$X^{(n)\mu}{}_\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{X_{22}^{(n)}(t, r, \theta, \phi)}{R^2(t, r)} & \frac{X_{23}^{(n)}(t, r, \theta, \phi)}{R^2(t, r)} \\ 0 & 0 & \frac{X_{23}^{(n)}(t, r, \theta, \phi)}{R^2(t, r) \sin^2 \theta} & \frac{X_{33}^{(n)}(t, r, \theta, \phi)}{R^2(t, r) \sin^2 \theta} \end{pmatrix}. \quad (\text{B.6})$$

On the other hand, from the eq. (B.1), the Bianchi identity reads

$$\nabla_\mu X^\mu{}_\nu = \epsilon^n \bar{\nabla}_\mu X^{(n)\mu}{}_\nu + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (\text{B.7})$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$. Then, zero-th and first components of this equation are

$$\nabla_\mu X^\mu{}_0 = -(X_{22}^{(n)} + (\sin \theta)^{-2} X_{33}^{(n)}) \frac{\partial_t R}{R^3} \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (\text{B.8})$$

$$\nabla_\mu X^\mu{}_1 = -(X_{22}^{(n)} + (\sin \theta)^{-2} X_{33}^{(n)}) \frac{\partial_r R}{R^3} \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (\text{B.9})$$

which yields the following solution when R is not a constant,

$$X_{22}^{(n)}(t, r, \theta, \phi) = -\frac{X_{33}^{(n)}(t, r, \theta, \phi)}{\sin^2 \theta}. \quad (\text{B.10})$$

The remaining components of this equation are given by

$$\nabla_\mu X^\mu{}_2 = (R \sin \theta)^{-2} \left(\partial_\phi X_{23}^{(n)} - \partial_\theta X_{33}^{(n)} \right) \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (\text{B.11})$$

$$\nabla_\mu X^\mu{}_3 = (R \sin \theta)^{-2} \left(\partial_\phi X_{33}^{(n)} + \sin \theta \partial_\theta (\sin \theta X_{23}^{(n)}) \right) \epsilon^n + \mathcal{O}(\epsilon^{n+1}) = 0, \quad (\text{B.12})$$

where we have used the relation (B.10). Removing $X_{23}^{(n)}$ from these equations leads to the following equation for $X_{33}^{(n)}$:

$$\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta X_{33}^{(n)}) + \frac{1}{\sin^2 \theta} \partial_\phi \partial_\phi X_{33}^{(n)} = 0. \quad (\text{B.13})$$

Since this is just the Laplace equation on a sphere, its solution is constant over the sphere:

$$X_{33}^{(n)} = f(t, r). \quad (\text{B.14})$$

By plugging this solution into eqs. (B.11) and (B.12), we obtain

$$X_{23}^{(n)} = \frac{g(t, r)}{\sin \theta}. \quad (\text{B.15})$$

Thus, the solution of the Bianchi identity is given by

$$X_{\mu\nu}^{(n)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{f(t, r)}{\sin^2 \theta} & \frac{g(t, r)}{\sin \theta} \\ 0 & 0 & \frac{g(t, r)}{\sin \theta} & f(t, r) \end{pmatrix}. \quad (\text{B.16})$$

However, the components with $\sin \theta$ in their denominators are singular at $\theta = 0, \pi$ unless

$$f(t, r) = 0, \quad (\text{B.17})$$

$$g(t, r) = 0. \quad (\text{B.18})$$

Therefore, the regular solution of $\nabla_\mu X^\mu{}_\nu = 0$ is

$$X_{\mu\nu}^{(n)} = 0. \quad (\text{B.19})$$

C Additional gauge symmetry of linear perturbations

The linear perturbations have an additional gauge symmetry, which is combination of gauge transformation of $g_{\mu\nu}$ and $f_{\mu\nu}$ separately but keeping the equation (4.11). In this appendix, we will give a concrete form of such coordinate transformation.

For this purpose, let us consider infinitesimal gauge transformation generated by $x^\mu \rightarrow x^\mu - \xi^\mu$ for $g_{\mu\nu}$ and $x^\mu \rightarrow x^\mu - (\xi^\mu + \delta \xi^\mu)$ for $f_{\mu\nu}$ ².

We denote the difference $A^2 \delta g_{ab} - \delta f_{ab}$ in eq. (4.11) under this transformation by Δ_{ab} , that is,

$$A^2 \delta g_{ab} - \delta f_{ab} \rightarrow A^2 \delta g_{ab} - \delta f_{ab} + \Delta_{ab}. \quad (\text{C.1})$$

²To determine the gauge transformation, one establish a bi-tangent bundle $T^2 M$ i.e. a fibre bundle locally isomorphic to $M \times T_p^{(g)} \times T_p^{(f)}$. To have a usual tangent bundle TM , two horizontal lifts $\pi_{(g)}^{-1}(M)$ and $\pi_{(f)}^{-1}(M)$ are identified by this relation of the diffeomorphisms so that it determines a diffeomorphism group of the base manifold.

The additional gauge symmetry is characterized by $\Delta_{ab} = 0$. The (2, 2) component of this condition is given by

$$0 = \frac{\Delta_{22}}{-2A^2R} = \delta\xi^0\partial_t R + \delta\xi^1\partial_r R + R\partial_\theta\delta\xi^2. \quad (\text{C.2})$$

The remaining (2, 3) and (3, 3) components are given by

$$0 = \frac{\Delta_{23}}{-A^2R^2\sin\theta} = \partial_\phi\left(\frac{\delta\xi^2}{\sin\theta}\right) + \sin\theta\partial_\theta\delta\xi^3, \quad (\text{C.3})$$

$$0 = \frac{\Delta_{33}}{-2A^2R^2\sin^3\theta} = -\partial_\theta\left(\frac{\delta\xi^2}{\sin\theta}\right) + \frac{\partial_\phi\delta\xi^3}{\sin\theta}, \quad (\text{C.4})$$

where we have used eq. (C.2). One can easily find, similarly to eq. (B.13), that these equations reduce to the Laplace equation on a sphere:

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\delta\xi^3) + \frac{1}{\sin^2\theta}\partial_\phi\partial_\phi\delta\xi^3 = 0, \quad (\text{C.5})$$

whose solution becomes

$$\delta\xi^3 = P(t, r). \quad (\text{C.6})$$

Plugging this solution into eqs. (C.3) and (C.4) we find

$$\delta\xi^2 = Q(t, r)\sin\theta. \quad (\text{C.7})$$

To sum up, this additional gauge symmetry is characterized by $\Xi(t, r, \theta, \phi)$, $P(t, r)$, $Q(t, r)$ as

$$\delta\xi^0 = \Xi(t, r, \theta, \phi), \quad (\text{C.8})$$

$$\delta\xi^1 = -\frac{\partial_t R(t, r)\Xi(t, r, \theta, \phi) + R(t, r)Q(t, r)\cos\theta}{\partial_r R(t, r)}, \quad (\text{C.9})$$

$$\delta\xi^2 = Q(t, r)\sin\theta, \quad (\text{C.10})$$

$$\delta\xi^3 = P(t, r). \quad (\text{C.11})$$

One may regard $R(t, r)$ itself as a radial coordinate and, in the new coordinates (t, R, θ, ϕ) , the above transformation (C.8)-(C.11) with $P(t, r) = Q(t, r) = 0$ simply reduces to the transformation of the time coordinate.

We can directly observe this symmetry in the action. Actually, the quadratic action of the mass term for the linear perturbations becomes

$$\begin{aligned} S_{\text{mass}}^{(2)} = & \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-\bar{g}} \frac{2C(t, r)}{AR^2} (\delta\gamma^2_2\delta\gamma^3_3 - \delta\gamma^2_3\delta\gamma^3_2) \\ & + \frac{M_{\text{pl}}^2}{2} \int d^4x \sqrt{-g}^{(2)} (-2\Lambda_{\text{eff}}^{(g)}) + \frac{\kappa^2 M_{\text{pl}}^2}{2} \int d^4x \sqrt{-f}^{(2)} (-2\Lambda_{\text{eff}}^{(f)}), \end{aligned} \quad (\text{C.12})$$

where $C(t, r)$ is the function defined in eq. (3.9), $\Lambda_{\text{eff}}^{(g)}$ and $\Lambda_{\text{eff}}^{(f)}$ are effective cosmological constants defined in eqs. (3.13) and (3.15), $\sqrt{-g}^{(2)}$ and $\sqrt{-f}^{(2)}$ are quadratic perturbations of $\sqrt{-g}$ and $\sqrt{-f}$. Clearly, this term is invariant under the transformation (C.8)-(C.11) because this transformation leaves $\delta\gamma^a_b$ unchanged.

D Hamiltonian analysis of linear perturbations

We will count the number of graviton degrees of freedom of linear perturbations by means of the Hamiltonian analysis. So, we omit the matter action in this appendix. For this purpose, it is useful to decompose the perturbations in terms of spherical harmonics Y_l^m as done in ref. [81]. Due to the spherical symmetry of the background metrics, the modes with different eigenvalues of rotation (l, m) or parity (odd or even) develop independently, and the dynamics of each mode does not depend on m . Hence, we may suppose that m is equal to zero, without loss of generality.

D.1 Odd mode perturbations

Non-vanishing components of the odd mode perturbations with $m = 0$ are given by

$$\delta g_{03} = \sum_{l \geq 1} h_0^{(g),l}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (\text{D.1})$$

$$\delta g_{13} = \sum_{l \geq 1} h_1^{(g),l}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (\text{D.2})$$

$$\delta g_{23} = \sum_{l \geq 2} h_2^{(g),l}(t, r) \sin^2 \theta \partial_\theta \left(\frac{\partial_\theta P_l(\cos \theta)}{\sin \theta} \right), \quad (\text{D.3})$$

and

$$\delta f_{03} = \sum_{l \geq 1} h_0^{(f),l}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (\text{D.4})$$

$$\delta f_{13} = \sum_{l \geq 1} h_1^{(f),l}(t, r) \sin \theta \partial_\theta P_l(\cos \theta), \quad (\text{D.5})$$

$$\delta f_{23} = \sum_{l \geq 2} h_2^{(f),l}(t, r) \sin^2 \theta \partial_\theta \left(\frac{\partial_\theta P_l(\cos \theta)}{\sin \theta} \right), \quad (\text{D.6})$$

where P_l is the Legendre polynomial. In this subsection, hereafter, we omit the suffix l and the summation with respect to l for brevity. From the perturbed Einstein-Hilbert action with the mass term (C.12), the conjugate momenta of $h_I^{(g/f)}$ ($I = 0, 1, 2$) are calculated as

$$P_0^{(g)} = \frac{\delta S}{\delta \dot{h}_0^{(g)}} = 0, \quad (\text{D.7})$$

$$P_1^{(g)} = \frac{\delta S}{\delta \dot{h}_1^{(g)}} = \frac{2\tilde{M}_{\text{pl}}^2}{\sqrt{-g}} \left(2(\ln R)' h_0^{(g)} - h_0^{(g)'} + \dot{h}_1^{(g)} \right), \quad (\text{D.8})$$

$$P_2^{(g)} = \frac{\delta S}{\delta \dot{h}_2^{(g)}} = \frac{2\lambda\tilde{M}_{\text{pl}}^2}{R^2} \sqrt{-g} \left(\bar{g}^{00} h_0^{(g)} + \bar{g}^{01} h_1^{(g)} - \bar{g}^{00} \dot{h}_2^{(g)} - \bar{g}^{01} h_2^{(g)'} \right), \quad (\text{D.9})$$

$$P_0^{(f)} = \frac{\delta S}{\delta \dot{h}_0^{(f)}} = 0, \quad (\text{D.10})$$

$$P_1^{(f)} = \frac{\delta S}{\delta \dot{h}_1^{(f)}} = \frac{2\kappa^2 \tilde{M}_{\text{pl}}^2}{\sqrt{-f}} \left(2(\ln R)' h_0^{(f)} - h_0^{(f)'} + \dot{h}_1^{(f)} \right), \quad (\text{D.11})$$

$$P_2^{(f)} = \frac{\delta S}{\delta \dot{h}_2^{(f)}} = \frac{2\lambda\kappa^2 \tilde{M}_{\text{pl}}^2}{A^2 R^2} \sqrt{-f} \left(\bar{f}^{00} h_0^{(f)} + \bar{f}^{01} h_1^{(f)} - \bar{f}^{00} \dot{h}_2^{(f)} - \bar{f}^{01} h_2^{(f)'} \right), \quad (\text{D.12})$$

where $\lambda := (l-1)(l+2)$, $\tilde{M}_{\text{pl}}^2 := \frac{l(1+l)}{1+2l} M_{\text{pl}}^2 \pi$, and $\sqrt{-\bar{g}}$ represents the determinant of only 0, 1 components:

$$\sqrt{-\bar{g}} := \sqrt{-\bar{g}_{00}\bar{g}_{11} + (\bar{g}_{01})^2}. \quad (\text{D.13})$$

We schematically decompose the Hamiltonian density as follows:

$$\mathcal{H}^{\text{odd}} = \mathcal{H}_{GR,(g)}^{\text{odd}} + \mathcal{H}_{GR,(f)}^{\text{odd}} + \mathcal{H}_{\text{mass}}^{\text{odd}}, \quad (\text{D.14})$$

where $\mathcal{H}_{GR,(g/f)}^{\text{odd}}$ represents the contribution from each Einstein-Hilbert term and the effective cosmological term, which is the second term (or third term) in the right hand side of eq. (C.12). $\mathcal{H}_{\text{mass}}^{\text{odd}}$ represents the contribution from the first term in the right hand side of eq. (C.12). This decomposition is justified because S_{mass} does not include time derivative of $g_{\mu\nu}$ and $f_{\mu\nu}$. From the expression of the action (C.12), $\mathcal{H}_{\text{mass}}^{\text{odd}}$ is explicitly calculated as

$$\mathcal{H}_{\text{mass}}^{\text{odd}} = \tilde{M}_{\text{pl}}^2 \lambda C(t, r) \frac{\sqrt{-\bar{g}}}{A^3 R^4} \left(A^2 h_2^{(g)} - h_2^{(f)} \right)^2. \quad (\text{D.15})$$

It should be noted that, for $l = 1$ mode, $\mathcal{H}_{\text{mass}}^{\text{odd}}$ vanishes, which implies that dynamics of $l = 1$ mode coincides with that of GR. Therefore, there should be additional gauge symmetry, under which each metric transforms independently. This transformation, actually, corresponds to the arbitrary function $P(t, r)$ in eq. (C.11).

From now on, we focus on $l \geq 2$ modes. The Hamiltonian density from the Einstein-Hilbert term is calculated as

$$\begin{aligned} \mathcal{H}_{GR,(g)}^{\text{odd}} = & \frac{1}{4\tilde{M}_{\text{pl}}^2} \sqrt{-\bar{g}} (P_1^{(g)})^2 + \frac{1}{4\tilde{M}_{\text{pl}}^2 \lambda} \frac{\sqrt{-\bar{g}} R^2}{\bar{g}_{11}} (P_2^{(g)})^2 + \frac{\bar{g}_{01}}{\bar{g}_{11}} (-h_1^{(g)} + h_2^{(g)})' P_2^{(g)} \\ & + \tilde{M}_{\text{pl}}^2 \lambda \frac{\sqrt{-\bar{g}}}{\bar{g}_{11} R^2} (h_2^{(g)})'^2 - 2\tilde{M}_{\text{pl}}^2 \lambda \frac{\sqrt{-\bar{g}}}{\bar{g}_{11} R^2} h_1^{(g)} h_2^{(g)'} + 4\tilde{M}_{\text{pl}}^2 \lambda \sqrt{-\bar{g}} \frac{\bar{g}^{1A} \partial_A (\ln R)'}{R^2} h_1^{(g)} h_2^{(g)} \\ & + M_1^{(g)} (h_1^{(g)})^2 + M_2^{(g)} (h_2^{(g)})^2 - h_0^{(g)} \mathcal{C}_{(g)}^{(1)} [P_1^{(g)}, P_2^{(g)}, h_1^{(g)}, h_2^{(g)}], \end{aligned} \quad (\text{D.16})$$

where $M_1^{(g)}, M_2^{(g)}$ are some functions of t, r and $\mathcal{C}_{(g)}^{(1)}$ is given by

$$\begin{aligned} \mathcal{C}_{(g)}^{(1)} = & 2(\ln R)' P_1^{(g)} + P_1^{(g)'} - P_2^{(g)} - 4\tilde{M}_{\text{pl}}^2 \frac{(\ln R)'}{\sqrt{-\bar{g}}} h_1^{(g)'} \\ & + 4\lambda \tilde{M}_{\text{pl}}^2 \sqrt{-\bar{g}} \bar{g}^{0A} \partial_A (\ln R) \frac{1}{R^2} h_2^{(g)} - 2\tilde{M}_{\text{pl}}^2 \frac{1}{R^2} \partial_r \left(\frac{(R^2)'}{\sqrt{-\bar{g}}} \right) h_1^{(g)}, \end{aligned} \quad (\text{D.17})$$

with $A = 0, 1$. $\mathcal{H}_{GR,(f)}^{\text{odd}}$ is obtained by replacing $g \rightarrow f$, $R \rightarrow AR$, $\tilde{M}_{\text{pl}}^2 \rightarrow \kappa^2 \tilde{M}_{\text{pl}}^2$.

The primary constraints of this system are

$$\mathcal{C}_{(g)}^{(0)} := P_0^{(g)} \approx 0, \quad (\text{D.18})$$

$$\mathcal{C}_{(f)}^{(0)} := P_0^{(f)} \approx 0, \quad (\text{D.19})$$

and then, the total Hamiltonian is

$$H_T^{\text{odd}} = H^{\text{odd}} + \int dr \left[v^{(g)}(t, r) \mathcal{C}_{(g)}^{(0)} + v^{(f)}(t, r) \mathcal{C}_{(f)}^{(0)} \right], \quad (\text{D.20})$$

$$H^{\text{odd}} = H_{GR,(g)}^{\text{odd}} + H_{GR,(f)}^{\text{odd}} + H_{\text{mass}}^{\text{odd}}, \quad (\text{D.21})$$

$$H_{GR,(g/f)}^{\text{odd}} = \int dr \mathcal{H}_{GR,(g/f)}^{\text{odd}}, \quad H_{\text{mass}}^{\text{odd}} = \int dr \mathcal{H}_{\text{mass}}^{\text{odd}}. \quad (\text{D.22})$$

Time evolution of the primary constraints is given by

$$\dot{\mathcal{C}}_{(g/f)}^{(0)} = \{\mathcal{C}_{(g/f)}^{(0)}, H_T^{\text{odd}}\} \approx \mathcal{C}_{(g/f)}^{(1)} [P_1^{(g/f)}, P_2^{(g/f)}, h_1^{(g/f)}, h_2^{(g/f)}], \quad (\text{D.23})$$

which generate the following two secondary constraints,

$$\mathcal{C}_{(g/f)}^{(1)} \approx 0. \quad (\text{D.24})$$

Time evolution of $\mathcal{C}_{(g)}^{(1)}$ is given by

$$\begin{aligned} \dot{\mathcal{C}}_{(g)}^{(1)} &= \frac{\partial \mathcal{C}_{(g)}^{(1)}}{\partial t} + \{\mathcal{C}_{(g)}^{(1)}, H_T^{\text{odd}}\} \approx -\{P_2^{(g)}, H_{\text{mass}}^{\text{odd}}\} \\ &= 2\lambda \tilde{M}_{\text{pl}}^2 \frac{\sqrt{-\bar{g}}}{AR^4} C(t, r) (A^2 h_2^{(g)} - h_2^{(f)}), \end{aligned} \quad (\text{D.25})$$

and that of $\mathcal{C}_{(f)}^{(1)}$ is given by

$$\dot{\mathcal{C}}_{(f)}^{(1)} \approx -2\lambda \tilde{M}_{\text{pl}}^2 \frac{\sqrt{-\bar{g}}}{A^3 R^4} C(t, r) (A^2 h_2^{(g)} - h_2^{(f)}). \quad (\text{D.26})$$

These equations impose another constraint,

$$\mathcal{C}^{(2)} := A^2 h_2^{(g)} - h_2^{(f)} \approx 0. \quad (\text{D.27})$$

From time evolution of $\mathcal{C}^{(2)}$,

$$\begin{aligned} \dot{\mathcal{C}}^{(2)} &= \{\mathcal{C}^{(2)}, H_T^{\text{odd}}\} \\ &\approx A^2 \{h_2^{(g)}, H_{GR, (g)}^{\text{odd}}\} - \{h_2^{(f)}, H_{GR, (f)}^{\text{odd}}\} \\ &\approx A^2 \left(h_0^{(g)} - \frac{\bar{g}_{01}}{\bar{g}_{11}} h_1^{(g)} + \frac{\sqrt{\bar{g}} R^2}{2\lambda \tilde{M}_{\text{pl}}^2 \bar{g}_{11}} P_2^{(g)} \right) \\ &\quad - \left(h_0^{(f)} - \frac{\bar{f}_{01}}{\bar{f}_{11}} h_1^{(f)} + \frac{\sqrt{\bar{f}} A^2 R^2}{2\lambda \kappa^2 \tilde{M}_{\text{pl}}^2 \bar{f}_{11}} P_2^{(f)} \right), \end{aligned} \quad (\text{D.28})$$

we obtain yet another constraint,

$$\begin{aligned} \mathcal{C}^{(3)} &:= A^2 \left(h_0^{(g)} - \frac{\bar{g}_{01}}{\bar{g}_{11}} h_1^{(g)} + \frac{\sqrt{\bar{g}} R^2}{2\lambda \tilde{M}_{\text{pl}}^2 \bar{g}_{11}} P_2^{(g)} \right) \\ &\quad - \left(h_0^{(f)} - \frac{\bar{f}_{01}}{\bar{f}_{11}} h_1^{(f)} + \frac{\sqrt{\bar{f}} A^2 R^2}{2\lambda \kappa^2 \tilde{M}_{\text{pl}}^2 \bar{f}_{11}} P_2^{(f)} \right). \end{aligned} \quad (\text{D.29})$$

Since $\mathcal{C}^{(3)}$ includes $h_0^{(g)}$ and $h_0^{(f)}$ terms, the Poisson brackets of $\mathcal{C}^{(3)}$ and primary constraints $\mathcal{C}_{(g/f)}^{(0)}$ do not vanish. Thus, the consistency relation on $\mathcal{C}^{(3)}$,

$$\dot{\mathcal{C}}^{(3)} \approx \partial_t \mathcal{C}^{(3)} + \{\mathcal{C}^{(3)}, H\} + A^2 v_{(g)} - v_{(f)} \approx 0, \quad (\text{D.30})$$

determines the combination of multipliers, $A^2 v_{(g)} - v_{(f)}$. Then, no further constraints are generated.

Since one multiplier remains undetermined, one can easily find that there is gauge symmetry in this system. More explicitly, one can confirm that there are two first class constraints (and four second class constraints) in this system through the presence of two zero eigenvalues of 6×6 matrix $\{\mathcal{C}_I, \mathcal{C}_J\}$, where \mathcal{C}_I represent all of the six constraints. These two gauge symmetries correspond to the ones which the theory originally possesses. To summarize, the number of graviton degrees of freedom in this system is

$$\frac{1}{2} \left(\overbrace{12}^{\text{variables}} - \overbrace{6}^{\text{constraints}} - \overbrace{2}^{\text{gauge dofs}} \right) = 2, \quad (\text{D.31})$$

and completely coincides with the case of two massless gravitons.

For the $l = 1$ mode, there are four variables (eight variables in phase space), $h_0^{(g/f)}, h_1^{(g/f)}$. As mentioned above, the interaction term S_{mass} vanishes for $l = 1$ mode, and hence the action reduces to decoupled two Einstein-Hilbert action. Then, there are four first class constraints and four gauge symmetries which correspond to the general covariance of $g_{\mu\nu}$ and $f_{\mu\nu}$ separately. These four gauge symmetries can be arranged into the ones of the full theory and the additional ones described by $P(t, r)$ in eq.(C.11). To summarize, the number of degrees of freedom of the odd $l = 1$ mode is

$$\frac{1}{2} \left(\overbrace{8}^{\text{variables}} - \overbrace{4}^{\text{constraints}} - \overbrace{4}^{\text{gauge dofs}} \right) = 0. \quad (\text{D.32})$$

D.2 Even mode perturbations

Similarly we consider even mode perturbations. Non-vanishing components of the even mode perturbations are given by

$$\delta g_{00} = \sum_{l \geq 0} H_0^{(g),l}(t, r) P_l(\cos \theta), \quad (\text{D.33})$$

$$\delta g_{01} = \sum_{l \geq 0} H_1^{(g),l}(t, r) P_l(\cos \theta), \quad (\text{D.34})$$

$$\delta g_{02} = \sum_{l \geq 1} H_2^{(g),l}(t, r) \partial_\theta P_l(\cos \theta), \quad (\text{D.35})$$

$$\delta g_{11} = \sum_{l \geq 0} H_3^{(g),l}(t, r) P_l(\cos \theta), \quad (\text{D.36})$$

$$\delta g_{12} = \sum_{l \geq 1} H_4^{(g),l}(t, r) \partial_\theta P_l(\cos \theta), \quad (\text{D.37})$$

$$\delta g_{22} = \sum_{l \geq 0} H_5^{(g),l}(t, r) P_l(\cos \theta) + \sum_{l \geq 2} H_6^{(g),l} \partial_\theta \partial_\theta P_l(\cos \theta), \quad (\text{D.38})$$

$$\delta g_{33} = \sum_{l \geq 0} H_5^{(g),l}(t, r) \sin^2 \theta P_l(\cos \theta) + \sum_{l \geq 2} H_6^{(g),l} \sin \theta \cos \theta \partial_\theta P_l(\cos \theta), \quad (\text{D.39})$$

and similar expansions are applied for $\delta f_{\mu\nu}$. In this subsection, hereafter, we omit the suffix l and the summation with respect to l for brevity. First we treat the $l \geq 2$ modes, and the Hamiltonian density for even modes is decomposed into

$$\mathcal{H}^{\text{even}} = \mathcal{H}_{GR,(g)}^{\text{even}} + \mathcal{H}_{GR,(f)}^{\text{even}} + \mathcal{H}_{\text{mass}}^{\text{even}}. \quad (\text{D.40})$$

$\mathcal{H}_{GR,(g/f)}^{\text{even}}$ represents a contribution from the Einstein-Hilbert term and effective cosmological constant terms, explicitly given by

$$\begin{aligned}\mathcal{H}_{GR,(g/f)}^{\text{even}} = & -H_0^{(g/f)} \mathcal{C}_{0,(g/f)}^{(1)} [P_3^{(g/f)}, P_5^{(g/f)}, H_3^{(g/f)}, H_4^{(g/f)}, H_5^{(g/f)}, H_6^{(g/f)}] \\ & -H_1^{(g/f)} \mathcal{C}_{1,(g/f)}^{(1)} [P_3^{(g/f)}, P_4^{(g/f)}, P_5^{(g/f)}, H_3^{(g/f)}, H_4^{(g/f)}, H_5^{(g/f)}, H_6^{(g/f)}] \\ & -H_2^{(g/f)} \mathcal{C}_{2,(g/f)}^{(1)} [P_4^{(g/f)}, P_6^{(g/f)}, H_3^{(g/f)}, H_4^{(g/f)}, H_5^{(g/f)}, H_6^{(g/f)}] \\ & + \left(\text{second order terms of } H_3^{(g/f)}, H_4^{(g/f)}, H_5^{(g/f)}, H_6^{(g/f)}, P_3^{(g/f)}, P_4^{(g/f)}, P_5^{(g/f)}, P_6^{(g/f)} \right)\end{aligned}\quad (\text{D.41})$$

with

$$\mathcal{C}_{0,(g)}^{(1)} = -R\bar{g}^{0I}(\partial_I R)P_5^{(g)} + \left(\text{linear terms of } P_3^{(g)}, H_3^{(g)}, H_4^{(g)}, H_5^{(g)}, H_6^{(g)} \right), \quad (\text{D.42})$$

$$\mathcal{C}_{1,(g)}^{(1)} = -2R\bar{g}^{1I}(\partial_I R)P_5^{(g)} + \left(\text{linear terms of } P_3^{(g)}, P_4^{(g)}, H_3^{(g)}, H_4^{(g)}, H_5^{(g)}, H_6^{(g)} \right), \quad (\text{D.43})$$

$$\mathcal{C}_{2,(g)}^{(1)} = -2P_6^{(g)} + \left(\text{linear terms of } P_4^{(g)}, H_3^{(g)}, H_4^{(g)}, H_5^{(g)}, H_6^{(g)} \right), \quad (\text{D.44})$$

where $P_I^{(g/f)}$ are the conjugate momenta of $H_I^{(g/f)}$. On the other hand, $\mathcal{H}_{\text{mass}}^{\text{even}}$ is given by

$$\begin{aligned}\mathcal{H}_{\text{mass}}^{\text{even}} = & -\hat{M}_{\text{pl}}^2 \frac{\sqrt{-\bar{g}}C(t,r)}{A^3 R^4} \left[(A^2 H_5^{(g)} - H_5^{(f)})^2 - l(l+1)(A^2 H_5^{(g)} - H_5^{(f)})(A^2 H_6^{(g)} - H_6^{(f)}) \right. \\ & \left. + \frac{l(l+1)}{2}(A^2 H_6^{(g)} - H_6^{(f)})^2 \right],\end{aligned}\quad (\text{D.45})$$

where $\hat{M}_{\text{pl}}^2 = M_{\text{pl}}^2 \pi / (1 + 2l)$. Then, the following six primary constraints are imposed,

$$\mathcal{C}_{I,(g/f)}^{(0)} := P_I^{(g/f)} \approx 0 \quad (\text{D.46})$$

for $I = 0, 1, 2$. The total Hamiltonian is

$$H_T^{\text{even}} = H^{\text{even}} + \int dr \left[v_{(g)}^I(t, r) P_I^{(g)} + v_{(f)}^I(t, r) P_I^{(f)} \right]. \quad (\text{D.47})$$

Time evolution of primary constraints is

$$\dot{\mathcal{C}}_{I,(g/f)}^{(0)} = \partial_t \mathcal{C}_{I,(g/f)}^{(0)} + \{\mathcal{C}_{I,(g/f)}^{(0)}, H_T\} \approx \{P_I^{(g/f)}, H_{GR,(g/f)}^{\text{even}}\} = \mathcal{C}_{I,(g/f)}^{(1)}, \quad (\text{D.48})$$

which impose six secondary constraints,

$$\mathcal{C}_{I,(g/f)}^{(1)} \approx 0. \quad (\text{D.49})$$

Time evolution of these constraints is given by

$$\dot{\mathcal{C}}_{0,(g)}^{(1)} = -2\hat{M}_{\text{pl}}^2 \frac{\sqrt{-\bar{g}}C(t,r)}{AR^3} \bar{g}^{0B} \partial_B R \left[(A^2 H_5^{(g)} - H_5^{(f)}) - \frac{l(l+1)}{2}(A^2 H_6^{(g)} - H_6^{(f)}) \right] \quad (\text{D.50})$$

$$\dot{\mathcal{C}}_{1,(g)}^{(1)} = -4\hat{M}_{\text{pl}}^2 \frac{\sqrt{-\bar{g}}C(t,r)}{AR^3} \bar{g}^{1B} \partial_B R \left[(A^2 H_5^{(g)} - H_5^{(f)}) - \frac{l(l+1)}{2}(A^2 H_6^{(g)} - H_6^{(f)}) \right] \quad (\text{D.51})$$

$$\dot{\mathcal{C}}_{2,(g)}^{(1)} = 2\hat{M}_{\text{pl}}^2 \frac{\sqrt{-\bar{g}}C(t,r)}{AR^4} l(l+1) \left[(A^2 H_5^{(g)} - H_5^{(f)}) - (A^2 H_6^{(g)} - H_6^{(f)}) \right], \quad (\text{D.52})$$

with $B = 0, 1$, and similar terms appear in the constraints for $f_{\mu\nu}$. Consequently, we obtain two additional constraints:

$$\mathcal{C}_1^{(2)} := A^2 H_5^{(g)} - H_5^{(f)}, \quad (\text{D.53})$$

$$\mathcal{C}_2^{(2)} := A^2 H_6^{(g)} - H_6^{(f)}. \quad (\text{D.54})$$

The time evolutions of these constraints are given by

$$\begin{aligned} \dot{\mathcal{C}}_1^{(2)} = & A^2 \left(Rg^{0B} \partial_B R H_0^{(g)} + 2Rg^{1B} \partial_B R H_1^{(g)} \right) - \left(A^2 R f^{0B} \partial_B R H_0^{(f)} + 2A^2 R f^{1B} \partial_B R H_1^{(f)} \right) \\ & + \text{linear terms of } H_4^{(g/f)}, H_5^{(g/f)}, H_6^{(g/f)}, P_3^{(g/f)}, P_5^{(g/f)}, P_6^{(g/f)}, \end{aligned} \quad (\text{D.55})$$

$$\dot{\mathcal{C}}_2^{(2)} = 2A^2 H_2^{(g)} - 2H_2^{(f)} + \text{linear terms of } H_4^{(g/f)}, H_5^{(g/f)}, H_6^{(g/f)}, P_5^{(g/f)}, P_6^{(g/f)}, \quad (\text{D.56})$$

which impose further two constraints,

$$\mathcal{C}_1^{(3)} := \dot{\mathcal{C}}_1^{(2)} \approx 0, \quad (\text{D.57})$$

$$\mathcal{C}_2^{(3)} := \dot{\mathcal{C}}_2^{(2)} \approx 0. \quad (\text{D.58})$$

Since the above constraints include $H_0^{(g/f)}, H_1^{(g/f)}, H_2^{(g/f)}$, time development of these constraints only determines two of the multipliers $v_{(g/f)}^I$ and hence no more constraint appears. One can see that four of the multipliers $v_{(g/f)}^I$ remain undetermined, which implies that this system has corresponding gauge symmetry. Concrete calculation shows that this system has eight first class constraints (and eight second class constraints) through eight non-zero eigenvalues of 16×16 matrix $\{\mathcal{C}_I, \mathcal{C}_J\}$, where \mathcal{C}_I represent all of the sixteen constraints. These eight constraints are composed of six gauge symmetry of full theory and two additional symmetry described by Ξ in eqs. (C.8) and (C.9). To summarize, the number of graviton degrees of freedom for even modes can be estimated as

$$\frac{1}{2} \left(\overbrace{28}^{\text{variables}} - \overbrace{16}^{\text{constraints}} - \overbrace{8}^{\text{gauge dofs}} \right) = 2, \quad (\text{D.59})$$

which again coincides with that of two massless gravitons for $l \geq 2$ modes.

The structure of Hamiltonian analysis is similar for $l = 0, 1$ mode. For $l = 0$ mode, initially we have eight variables (sixteen phase space variables), $H_0^{(g/f)}, H_1^{(g/f)}, H_3^{(g/f)}, H_5^{(g/f)}$. Similar analysis shows that there are ten constraints, $\mathcal{C}_{0,(g/f)}^{(0)}, \mathcal{C}_{1,(g/f)}^{(0)}, \mathcal{C}_{0,(g/f)}^{(1)}, \mathcal{C}_{1,(g/f)}^{(1)}, \mathcal{C}_1^{(2)}, \mathcal{C}_1^{(3)}$ and six gauge degrees of freedom, that is, there are six first class constraints and four second class constraints. Four gauge degrees of freedom come from the ones of the full theory and two come from the additional ones described by Ξ in eqs. (C.8), (C.9). Then, the number of dynamical degrees of freedom is

$$\frac{1}{2} \left(\overbrace{16}^{\text{variables}} - \overbrace{10}^{\text{constraints}} - \overbrace{6}^{\text{gauge dofs}} \right) = 0. \quad (\text{D.60})$$

For $l = 1$ mode, we have twelve variables (twenty-four phase space variables), $H_0^{(g/f)}, H_1^{(g/f)}, H_2^{(g/f)}, H_3^{(g/f)}, H_4^{(g/f)}, H_5^{(g/f)}$, fourteen constraints, $\mathcal{C}_{0,(g/f)}^{(0)}, \mathcal{C}_{1,(g/f)}^{(0)}, \mathcal{C}_{2,(g/f)}^{(0)}, \mathcal{C}_{0,(g/f)}^{(1)}, \mathcal{C}_{1,(g/f)}^{(1)}$,

$\mathcal{C}_{2,(g/f)}^{(1)}, \mathcal{C}_1^{(2)}, \mathcal{C}_1^{(3)}$ and ten gauge degrees of freedom, that is, there are ten first class constraints and four second class constraints. Then, the number of dynamical degrees of freedom is

$$\frac{1}{2} \left(\overbrace{24}^{\text{variables}} - \overbrace{14}^{\text{constraints}} - \overbrace{10}^{\text{gauge dofs}} \right) = 0. \quad (\text{D.61})$$

It should be noted that six gauge symmetries correspond to the one of full theory, two gauge symmetries correspond to Ξ in eqs. (C.8),(C.9), and the other two gauge symmetries correspond to $Q(t, r)$ in eq. (C.10).

References

- [1] M. Fierz and W. Pauli, *On relativistic wave equations for particles of arbitrary spin in an electromagnetic field*, *Proc.Roy.Soc.Lond.* **A173** (1939) 211–232.
- [2] H. van Dam and M. Veltman, *Massive and massless Yang-Mills and gravitational fields*, *Nucl.Phys.* **B22** (1970) 397–411.
- [3] V. Zakharov, *Linearized gravitation theory and the graviton mass*, *JETP Lett.* **12** (1970) 312.
- [4] A. Vainshtein, *To the problem of nonvanishing gravitation mass*, *Phys.Lett.* **B39** (1972) 393–394.
- [5] D. Boulware and S. Deser, *Can gravitation have a finite range?*, *Phys.Rev.* **D6** (1972) 3368–3382.
- [6] C. de Rham and G. Gabadadze, *Generalization of the Fierz-Pauli Action*, *Phys.Rev.* **D82** (2010) 044020, [[arXiv:1007.0443](#)].
- [7] C. de Rham, G. Gabadadze, and A. J. Tolley, *Resummation of Massive Gravity*, *Phys.Rev.Lett.* **106** (2011) 231101, [[arXiv:1011.1232](#)].
- [8] S. Hassan and R. A. Rosen, *Resolving the Ghost Problem in non-Linear Massive Gravity*, *Phys.Rev.Lett.* **108** (2012) 041101, [[arXiv:1106.3344](#)].
- [9] S. Hassan, R. A. Rosen, and A. Schmidt-May, *Ghost-free Massive Gravity with a General Reference Metric*, *JHEP* **1202** (2012) 026, [[arXiv:1109.3230](#)].
- [10] S. Hassan and R. A. Rosen, *Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity*, *JHEP* **1204** (2012) 123, [[arXiv:1111.2070](#)].
- [11] C. de Rham, G. Gabadadze, and A. J. Tolley, *Ghost free Massive Gravity in the Stückelberg language*, *Phys.Lett.* **B711** (2012) 190–195, [[arXiv:1107.3820](#)].
- [12] M. Mirbabayi, *A Proof Of Ghost Freedom In de Rham-Gabadadze-Tolley Massive Gravity*, *Phys.Rev.* **D86** (2012) 084006, [[arXiv:1112.1435](#)].
- [13] T. Kugo and N. Ohta, *Covariant Approach to the No-ghost Theorem in Massive Gravity*, *PTEP* **2014** (2014), no. 4 043B04, [[arXiv:1401.3873](#)].
- [14] X. Gao, T. Kobayashi, M. Yamaguchi, and D. Yoshida, *Covariant Stückelberg analysis of dRGT massive gravity with a general fiducial metric*, *Phys.Rev.* **D90** (2014) 124073, [[arXiv:1409.3074](#)].
- [15] S. Hassan and R. A. Rosen, *Bimetric Gravity from Ghost-free Massive Gravity*, *JHEP* **1202** (2012) 126, [[arXiv:1109.3515](#)].
- [16] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, *Open FRW universes and self-acceleration from nonlinear massive gravity*, *JCAP* **1111** (2011) 030, [[arXiv:1109.3845](#)].

- [17] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, *Cosmological perturbations of self-accelerating universe in nonlinear massive gravity*, *JCAP* **1203** (2012) 006, [[arXiv:1111.4107](#)].
- [18] A. De Felice, A. E. Gumrukcuoglu, and S. Mukohyama, *Massive gravity: nonlinear instability of the homogeneous and isotropic universe*, *Phys.Rev.Lett.* **109** (2012) 171101, [[arXiv:1206.2080](#)].
- [19] S. H. Pereira, E. L. Mendonca, A. P. S. S., and J. F. Jesus, *Cosmological bounds on open FLRW solutions of massive gravity*, [arXiv:1504.02295](#).
- [20] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, *Anisotropic Friedmann-Robertson-Walker universe from nonlinear massive gravity*, *Phys.Lett.* **B717** (2012) 295–298, [[arXiv:1206.2723](#)].
- [21] A. E. G. De Felice, Antonio and, C. Lin, and S. Mukohyama, *Nonlinear stability of cosmological solutions in massive gravity*, *JCAP* **1305** (2013) 035, [[arXiv:1303.4154](#)].
- [22] A. De Felice, A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, *On the cosmology of massive gravity*, *Class. Quant. Grav.* **30** (2013) 184004, [[arXiv:1304.0484](#)].
- [23] K. Koyama, G. Niz, and G. Tasinato, *Analytic solutions in non-linear massive gravity*, *Phys.Rev.Lett.* **107** (2011) 131101, [[arXiv:1103.4708](#)].
- [24] G. D’Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, et al., *Massive Cosmologies*, *Phys.Rev.* **D84** (2011) 124046, [[arXiv:1108.5231](#)].
- [25] P. Gratia, W. Hu, and M. Wyman, *Self-accelerating Massive Gravity: Exact solutions for any isotropic matter distribution*, *Phys.Rev.* **D86** (2012) 061504, [[arXiv:1205.4241](#)].
- [26] A. H. Chamseddine and M. S. Volkov, *Cosmological solutions with massive gravitons*, *Phys.Lett.* **B704** (2011) 652–654, [[arXiv:1107.5504](#)].
- [27] T. Kobayashi, M. Siino, M. Yamaguchi, and D. Yoshida, *New Cosmological Solutions in Massive Gravity*, *Phys.Rev.* **D86** (2012) 061505, [[arXiv:1205.4938](#)].
- [28] G. D’Amico, *Cosmology and perturbations in massive gravity*, *Phys.Rev.* **D86** (2012) 124019, [[arXiv:1206.3617](#)].
- [29] M. Wyman, W. Hu, and P. Gratia, *Self-accelerating Massive Gravity: Time for Field Fluctuations*, *Phys.Rev.* **D87** (2013), no. 8 084046, [[arXiv:1211.4576](#)].
- [30] N. Khosravi, G. Niz, K. Koyama, and G. Tasinato, *Stability of the Self-accelerating Universe in Massive Gravity*, *JCAP* **1308** (2013) 044, [[arXiv:1305.4950](#)].
- [31] P. Motloch and W. Hu, *Self-accelerating Massive Gravity: Covariant Perturbation Theory*, *Phys.Rev.* **D90** (2014), no. 10 104027, [[arXiv:1409.2204](#)].
- [32] D. Langlois and A. Naruko, *Cosmological solutions of massive gravity on de Sitter*, *Class.Quant.Grav.* **29** (2012) 202001, [[arXiv:1206.6810](#)].
- [33] D. Langlois and A. Naruko, *Bouncing cosmologies in massive gravity on de Sitter*, *Class.Quant.Grav.* **30** (2013) 205012, [[arXiv:1305.6346](#)].
- [34] M. Fasiello and A. J. Tolley, *Cosmological perturbations in Massive Gravity and the Higuchi bound*, *JCAP* **1211** (2012) 035, [[arXiv:1206.3852](#)].
- [35] S. Pan and S. Chakraborty, *A Cosmological Study in Massive Gravity theory*, *Annals Phys.* **360** (2015) 180–193, [[arXiv:1505.00743](#)].
- [36] M. S. Volkov, *Cosmological solutions with massive gravitons in the bigravity theory*, *JHEP* **1201** (2012) 035, [[arXiv:1110.6153](#)].
- [37] M. S. Volkov, *Exact self-accelerating cosmologies in the ghost-free bigravity and massive gravity*, *Phys.Rev.* **D86** (2012) 061502, [[arXiv:1205.5713](#)].

- [38] M. S. Volkov, *Exact self-accelerating cosmologies in the ghost-free massive gravity – the detailed derivation*, *Phys.Rev.* **D86** (2012) 104022, [[arXiv:1207.3723](#)].
- [39] M. S. Volkov, *Self-accelerating cosmologies and hairy black holes in ghost-free bigravity and massive gravity*, *Class.Quant.Grav.* **30** (2013) 184009, [[arXiv:1304.0238](#)].
- [40] M. von Strauss, A. Schmidt-May, J. Enander, E. Mortsell, and S. Hassan, *Cosmological Solutions in Bimetric Gravity and their Observational Tests*, *JCAP* **1203** (2012) 042, [[arXiv:1111.1655](#)].
- [41] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo, *FRW Cosmology in Ghost Free Massive Gravity*, *JHEP* **1203** (2012) 067, [[arXiv:1111.1983](#)].
- [42] F. Koennig, A. Patil, and L. Amendola, *Viable cosmological solutions in massive bimetric gravity*, *JCAP* **1403** (2014) 029, [[arXiv:1312.3208](#)].
- [43] D. Comelli, M. Crisostomi, and L. Pilo, *Perturbations in Massive Gravity Cosmology*, *JHEP* **1206** (2012) 085, [[arXiv:1202.1986](#)].
- [44] M. Berg, I. Buchberger, J. Enander, E. Mortsell, and S. Sjors, *Growth Histories in Bimetric Massive Gravity*, *JCAP* **1212** (2012) 021, [[arXiv:1206.3496](#)].
- [45] N. Khosravi, H. R. Sepangi, and S. Shahidi, *Massive cosmological scalar perturbations*, *Phys.Rev.* **D86** (2012) 043517, [[arXiv:1202.2767](#)].
- [46] D. Comelli, M. Crisostomi, and L. Pilo, *FRW Cosmological Perturbations in Massive Bigravity*, *Phys.Rev.* **D90** (2014), no. 8 084003, [[arXiv:1403.5679](#)].
- [47] F. Koennig and L. Amendola, *Instability in a minimal bimetric gravity model*, *Phys.Rev.* **D90** (2014), no. 4 044030, [[arXiv:1402.1988](#)].
- [48] A. De Felice, A. E. Gumrukcuoglu, S. Mukohyama, N. Tanahashi, and T. Tanaka, *Viable cosmology in bimetric theory*, *JCAP* **1406** (2014) 037, [[arXiv:1404.0008](#)].
- [49] A. R. Solomon, Y. Akrami, and T. S. Koivisto, *Linear growth of structure in massive bigravity*, *JCAP* **1410** (2014), no. 10 066, [[arXiv:1404.4061](#)].
- [50] F. Koennig, Y. Akrami, L. Amendola, M. Motta, and A. R. Solomon, *Stable and unstable cosmological models in bimetric massive gravity*, *Phys.Rev.* **D90** (2014), no. 12 124014, [[arXiv:1407.4331](#)].
- [51] M. Lagos and P. G. Ferreira, *Cosmological perturbations in massive bigravity*, *JCAP* **1412** (2014), no. 12 026, [[arXiv:1410.0207](#)].
- [52] G. Cusin, R. Durrer, P. Guarato, and M. Motta, *Gravitational waves in bigravity cosmology*, [[arXiv:1412.5979](#)].
- [53] J. Enander, Y. Akrami, E. Mortsell, M. Renneby, and A. R. Solomon, *Integrated Sachs-Wolfe effect in massive bigravity*, [[arXiv:1501.02140](#)].
- [54] Y. Akrami, S. F. Hassan, F. Koennig, A. Schmidt-May, and A. R. Solomon, *Bimetric gravity is cosmologically viable*, *Phys. Lett.* **B748** (2015) 37–44, [[arXiv:1503.07521](#)].
- [55] M. Fasiello and R. H. Ribeiro, *Mild bounds on bigravity from primordial gravitational waves*, *JCAP* **1507** (2015), no. 07 027, [[arXiv:1505.00404](#)].
- [56] C. de Rham, L. Heisenberg, and R. H. Ribeiro, *On couplings to matter in massive (bi-)gravity*, *Class. Quant. Grav.* **32** (2015) 035022, [[arXiv:1408.1678](#)].
- [57] C. de Rham, L. Heisenberg, and R. H. Ribeiro, *Ghosts and matter couplings in massive gravity, bigravity and multigravity*, *Phys. Rev.* **D90** (2014) 124042, [[arXiv:1409.3834](#)].
- [58] A. Emir Gumrukcuoglu, L. Heisenberg, and S. Mukohyama, *Cosmological perturbations in massive gravity with doubly coupled matter*, *JCAP* **1502** (2015), no. 02 022, [[arXiv:1409.7260](#)].

- [59] A. R. Solomon, J. Enander, Y. Akrami, T. S. Koivisto, F. Konnig, et al., *Does massive gravity have viable cosmologies?*, [arXiv:1409.8300](#).
- [60] X. Gao and D. Yoshida, *On coupling between Galileon and massive gravity with composite metrics*, [arXiv:1412.8471](#).
- [61] A. E. Gumrukcuoglu, L. Heisenberg, S. Mukohyama, and N. Tanahashi, *Cosmology in bimetric theory with an effective composite coupling to matter*, [arXiv:1501.02790](#).
- [62] D. Comelli, M. Crisostomi, K. Koyama, L. Pilo, and G. Tasinato, *Cosmology of bigravity with doubly coupled matter*, [arXiv:1501.00864](#).
- [63] L. Heisenberg, *Non-minimal derivative couplings of the composite metric*, [arXiv:1506.00580](#).
- [64] M. S. Volkov, *Hairy black holes in the ghost-free bigravity theory*, *Phys.Rev.* **D85** (2012) 124043, [[arXiv:1202.6682](#)].
- [65] M. S. Volkov, *Hairy black holes in theories with massive gravitons*, *Lect.Notes Phys.* **892** (2015) 161–180, [[arXiv:1405.1742](#)].
- [66] D. Comelli, M. Crisostomi, F. Nesti, and L. Pilo, *Spherically Symmetric Solutions in Ghost-Free Massive Gravity*, *Phys.Rev.* **D85** (2012) 024044, [[arXiv:1110.4967](#)].
- [67] T. Katsuragawa, *Properties of Bigravity Solutions in a Solvable Class*, *Phys.Rev.* **D89** (2014) 124007, [[arXiv:1312.1550](#)].
- [68] S. G. Ghosh, L. Tannukij, and P. Wongjun, *A class of black holes in dRGT massive gravity and their thermodynamical properties*, [arXiv:1506.07119](#).
- [69] E. Babichev and R. Brito, *Black holes in massive gravity*, *Class. Quant. Grav.* **32** (2015), no. 15 154001, [[arXiv:1503.07529](#)].
- [70] E. Babichev and A. Fabbri, *Instability of black holes in massive gravity*, *Class.Quant.Grav.* **30** (2013) 152001, [[arXiv:1304.5992](#)].
- [71] R. Brito, V. Cardoso, and P. Pani, *Massive spin-2 fields on black hole spacetimes: Instability of the Schwarzschild and Kerr solutions and bounds on the graviton mass*, *Phys.Rev.* **D88** (2013), no. 2 023514, [[arXiv:1304.6725](#)].
- [72] R. Brito, V. Cardoso, and P. Pani, *Partially massless gravitons do not destroy general relativity black holes*, *Phys.Rev.* **D87** (2013), no. 12 124024, [[arXiv:1306.0908](#)].
- [73] T. Katsuragawa and S. Nojiri, *Stability and Anti-evaporation of the Schwarzschild-de Sitter Black Holes in Bigravity*, [arXiv:1411.1610](#).
- [74] K. Koyama, G. Niz, and G. Tasinato, *Strong interactions and exact solutions in non-linear massive gravity*, *Phys.Rev.* **D84** (2011) 064033, [[arXiv:1104.2143](#)].
- [75] E. Babichev and A. Fabbri, *Stability analysis of black holes in massive gravity: a unified treatment*, *Phys.Rev.* **D89** (2014) 081502, [[arXiv:1401.6871](#)].
- [76] T. Nieuwenhuizen, *Exact Schwarzschild-de Sitter black holes in a family of massive gravity models*, *Phys.Rev.* **D84** (2011) 024038, [[arXiv:1103.5912](#)].
- [77] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, and A. Tolley, *On Black Holes in Massive Gravity*, *Phys.Rev.* **D85** (2012) 044024, [[arXiv:1111.3613](#)].
- [78] I. Arraut, *On the Black Holes in alternative theories of gravity: The case of non-linear massive gravity*, *Int.J.Mod.Phys.* **D24** (2015) 1550022, [[arXiv:1311.0732](#)].
- [79] H. Kodama and I. Arraut, *Stability of the Schwarzschild de Sitter black hole in the dRGT massive gravity theory*, *PTEP* **2014** (2014), no. 2 023E02, [[arXiv:1312.0370](#)].
- [80] L. Bernard, C. Deffayet, and M. von Strauss, *Consistent massive graviton on arbitrary background*, [arXiv:1410.8302](#).

- [81] T. Regge and J. A. Wheeler, *Stability of a Schwarzschild singularity*, *Phys.Rev.* **108** (1957) 1063–1069.